On Existence and Asymptotic Behavior of the Solution for a Fractional Integro-Differential Equation

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Abstract

In this paper, we have taken a fractional integro-differential equation of Volterra nonlinear type integral equation with initial conditions, and we have studied the existence and asymptotic behavior of solution in the space of p-integrable functions on $(0,b)$, $L^p(0,b)$, $1 < p < \infty$. Our method is based on the applications of the Banach fixed point theorem, generalized Gronwall's lemma and Hölder's inequality.

Keywords: Asymptotic behavior of solution, Banach fixed point theorem, generalized Gronwall's lemma and Hölder's inequality

1 Introduction

In this paper, we consider the existence and asymptotic behavior of solution of the following fractional integro-differential equation

$$I^{-\alpha} y(x) = f(x, y(x)) + \int_0^x g(x, t, y(t))dt \qquad ,0 < x < b \;, n-1 < \alpha \le n, \;\; n \ge 2, \qquad (1.1)$$

with the initial conditions:

$$y^{(i)}(0) = l_{i+1} \qquad l_{i+1} \in R, \qquad \text{for} \quad i = 0,1,2,\ldots,n-1, \qquad (1.2)$$

Where $f : 0,b \times R \to R$ and $g : 0,b \times 0,b \times R \to R$ are continuous functions, R is the set of real numbers. Fractional integro-differential equations play an important role in modeling processes in applied mathematics in various fields (physics, engineering, finance, biology....). There are many real life problems that can be modeled by fractional differential equations in acoustics, electromagnetic, diffusion processes, viscoelasticity, hydrology, heat conduction in materials with memory, and other areas, see [2, 5, 6] for more details.

Nowadays, fractional integro-differential equation has gained great authors' attention, because of its wide range of applications. Some existence and asymptotic behavior of solutions of fractional integro-differential equations recent works considered by Momani S. et al [3], [4]. Wu J. et al [8], Ahmad B. et al [1].

2 Preliminaries

First, we shall set forth some preliminaries and hypotheses that will be used in our discussion, for details see [2, 7].

Definition 2.1. . Let *f* be a function which is defined almost everywhere (a.e) on a,b . For $\alpha > 0$,we define

$$ {}_a^x I^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^x x - t^{\alpha - 1} f(t) dt $$

provided that this integral (Lebsegue) exists, where Γ is gamma function. where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Definition 2.2. For a function f given on the interval $[a,b]$, the Caputo fractional derivative of order $\alpha > 0$, of f is defined by

$$ {}_a^x I_c^{-\alpha} f = \frac{1}{\Gamma(n-\alpha)} \int_a^x f^{(n)}(t)\, x - t^{n-\alpha-1} dt $$

Lemma 2.1. Let $\alpha > 0$, then:

$$ {}_a^x I^{\alpha}\, {}_a^t I_c^{-\alpha} f = f(x) + c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1} $$

for some $c_i \in R$, *i=0, 1, 2, ..., n-1*, $n = [\alpha] + 1$.

Lemma 2.2. If $\alpha > 0$ and $f \in L(a,b)$, then ${}_a^x I^\alpha f$ exists everywhere if $\alpha \geq 1$ and (a.e.) if $\alpha < 1$. for all $x \in [a,b]$.

Lemma 2.3. If $\alpha \geq 1$ and $f(x) \in L(a,b)$, then ${}_a^x I^\alpha f$ is absolutely continuous in $x \in [a,b]$.

Lemma 2.4. (Hölder's inequality)

Let X be a measurable space, let p and q satisfy $1 < p < \infty$, $1 < q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(X)$ and $g \in L^q(X)$, then $(f\,g)$ belongs to $L(X)$ and satisfies

$$\int_X |f\,g|\,dx \leq \left[\int_X |f|^p dx \right]^{\frac{1}{p}} \left[\int_X |g|^q dx \right]^{\frac{1}{q}}.$$

Lemma 2.5. (**Generalized Gronwall's Lemma**).

Let $w(x)$, $v(x)$ and $y(x)$ be continuous functions in a subinterval Ω of R. Assume that y and v are continuous and w is a non-decreasing that its negative part is integrable on every closed and bounded subinterval of Ω. If y is non-negative and if v satisfies the integral inequality:

$$y(x) \leq w(x) + \int_{x_0}^{x} v(t)\,y(t)\,dt \qquad \text{, for all } x \geq x_0$$

then

$$y(x) \leq w(x) \exp\left(\int_{x_0}^{x} v(t)\,dt \right) \qquad \text{, for all } x \geq x_0 .$$

Theorem 2.1. If there exists a Lebesgue integrable function g on a,b such that $|f| \leq g$ a.e. on a,b where f is measurable, then f Lebesgue integrable function.

3 The Main Results.

In this section, we shall prove existence of solution in $L^p(0,b)$, also we study the asymptotic behavior of solutions in $L^p(0,b)$ as $b \to \infty$, for the fractional integro-differential equation (1.1) satisfying (1.2).

Consider $L^p(0,b)$ to be the space of all measurable functions f such that $|f|^p$ is Lebesgue integrable on $(0,b)$. For any $f \in L^p(0,b)$, we define the norm as $\|f\| = \left(\int_0^b |f|^p dx\right)^{1/p}$, under this norm, the space $L^p(0,b)$ is a Banach space.

Lemma 3.1. Let $n-1<\alpha\le n$ and $f : [0,b] \times R \mapsto R$ be continuous. A function y is a solution of fractional integral equation

$$y(x)=\sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t,y(t))dt+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))ds\,dt \qquad (3.1)$$

if and only if it is a solution of a fractional integro-differential equation (1.1) satisfying (1.2).

Proof. Operate both sides of equation (1.1) by the operator ${}_0^xI^{\alpha}$ we obtain

$$ {}_0^xI^{\alpha}\,{}_0^tI^{-\alpha}y(x)={}_0^xI^{\alpha}f(x,y(x)) + {}_0^xI^{\alpha}\int_0^x g(x,t,y(t))\,dt$$

by using Lemma 2.1, we get

$$y(x)=c_0+c_1x+c_2x^2+\cdots+c_{n-1}x^{n-1}+{}_0^xI^{\alpha}f(x,y(x)) + {}_0^xI^{\alpha}\int_0^x g(x,t,y(t))\,dt \qquad (3.2)$$

$$y'(x)=c_1+2c_2x+\cdots+(n-1)\,c_{n-1}x^{n-2}+{}_0^xI^{\alpha-1}f(x,y(x)) + {}_0^xI^{\alpha-1}\int_0^x g(x,t,y(t))\,dt$$

$$y''(x)=2c_2+\cdots+(n-1)(n-2)\,c_{n-1}x^{n-3}+{}_0^xI^{\alpha-2}f(x,y(x)) + {}_0^xI^{\alpha-2}\int_0^x g(x,t,y(t))\,dt$$

$$\vdots$$

$$y^{(n-1)}(x)=(n-1)(n-2)(n-3)\ldots 2\,c_{n-1}+{}_0^xI^{\alpha-n}f(x,y(x)) + {}_0^xI^{\alpha-n}\int_0^x g(x,t,y(t))\,dt,$$

using the initial conditions (1.2), we obtain

$$c_0=l_1\ ,\ c_1=\frac{1}{1!}l_2\ ,\ c_2=\frac{1}{2!}l_3\ ,c_3=\frac{1}{3!}l_4\ ,\ldots,c_{n-1}=\frac{1}{(n-1)!}l_n \qquad (3.3)$$

substituting equation (3.3) in equation (3.2), we obtain the final form of $y(x)$ as equation (3.1).

Theorem 3.1: Let the right hand side of the fractional integro-differential equation (1.1) satisfy the following conditions:

i- $f(x,y(x))$, $\int_0^x g(x,t,y(t))dt \in L^p[0,b]$,

ii- $|f(x,y_2(x))-f(x,y_1(x))| \le Z(x)|y_2-y_1|$ and $\int_0^x |g(x,t,y_2(t))-g(x,t,y_1(t))|dt \le P(x)|y_2-y_1|$

for all $x \in (0,b)$ and $y_1, y_2 \in L^p[0,b]$, where $Z(x)$ and $P(x)$ are non – negative continuous and bounded functions on $[0,b]$, then if:

$$\frac{2\,b^{\alpha}(k_1+k_2)}{\Gamma(\alpha)}\sqrt[p]{\frac{(p-1)^{p-1}}{p\alpha\,(p\alpha-1)^{p-1}}} < 1 \tag{3.4}$$

there exists a p–integrable solution of the fractional integro-differential equation (1.1) satisfying (1.2).

Proof. Let the mapping T on $L^p[0,b]$ be defined as

$$T\,y(x) = \sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t,y(t))dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))ds\,dt \tag{3.5}$$

we have to prove that T maps $L^p[0,b]$ into itself. Let:

$$h(x) = \sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t,y(t))dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))ds\,dt$$

$$|h(x)|^p = \left|\sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t,y(t))dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))ds\,dt\right|^p$$

$$|h(x)|^p \le 2^p\left|\sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!}\right|^p + \frac{2^p}{(\Gamma(\alpha))^p}\left|\int_0^x (x-t)^{\alpha-1} f(t,y(t))\,dt\right|^p + \frac{2^p}{(\Gamma(\alpha))^p}\left|\int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))\,ds\,dt\right|^p$$

let $h_1(x) = \sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!}$, $h_2(x) = \int_0^x (x-t)^{\alpha-1} f(t,y(t))dt$ and $h_3(x) = \int_0^x (x-t)^{\alpha-1}\int_0^t g(t,s,y(s))ds\,dt$.

$h_1(x)$ is continuous , thus it is measurable, hence $|h_1(x)|^p$ is measurable and it is Lebesgue integrable.

Now we take $$|h_2(x)|^p = \left|\int_0^x (x-t)^{\alpha-1} f(t, y(t))\,dt\right|^p \tag{3.6}$$

by Lemma 2.2 and Lemma 2.3 $h_2(x)$ exists and it is absolutely continuous and so it is continuous. Thus $h_2(x)$ is measurable and hence $|h_2(x)|^p$ is measurable. Then we must show that $|h_2(x)|^p$ is Lebesgue integrable. Since by condition (i) $f(x, y(x)) \in L^p(0,b)$ and we have $(x-t)^{\alpha-1} \in L^q(0,b)$, then by Hölder's inequality and from equation (3.6) we obtain

$$|h_2(x)|^p \le \left[\left(\int_0^x (x-t)^{q\,\alpha-1}\,dt\right)^{\frac{1}{q}} \left(\int_0^x |f(t, y(t))|^p\,dt\right)^{\frac{1}{p}}\right]^p$$

$$\le \left(\frac{x^{\frac{p}{p-1}\,\alpha-1\,+1}}{\frac{p}{p-1}\,\alpha-1\,+1}\right)^{p-1} \int_0^x |f(t, y(t))|^p\,dt$$

$$\le \frac{x^{p\alpha-1}}{\left(\frac{p\alpha-1}{p-1}\right)^{p-1}} \int_0^x |f(t, y(t))|^p\,dt$$

from definition (2.1), for $0 \le x \le b$, we have:

$$\int_0^x\int_0^t |f(t, y(t))|^p ds\,dt = \int_0^x \; x-t \;\; |f(t, y(t))|^p\,dt \;=\; {}_0^xI^2 |f(t, y(t))|^p .$$

Thus by Lemma 2.2, it follows that $\int_0^x |f(t, y(t))|^p dt$ is Lebesgue integrable for all $x \in (0,b)$, hence by Theorem 2.1 $|h_2(x)|^p$ is Lebesgue integrable. Now we take

$$|h_3(x)|^p = \left|\int_0^x (x-t)^{\alpha-1} \int_0^t g(t, s, y(s))\,ds\,dt\right|^p \tag{3.7}$$

By condition (i), $\int_o^x g(x, t, y(t))\,dt \;\in L^p\;\; 0,b$, and we have $(x-t)^{\alpha-1} \in L^q(0,b)$ then by Hölder's inequality and from equation (3.7) we obtain

$$|h_3(x)|^p \le \left[\left(\int_0^x (x-t)^{q(\alpha-1)}\, dt\right)^{\frac{1}{q}} \left(\int_0^x \left(\int_0^t |g(t,s,y(s))|\,ds\right)^p dt\right)^{\frac{1}{p}}\right]^p$$

$$\le \frac{x^{p\alpha-1}}{\left(\frac{p\alpha-1}{p-1}\right)^{p-1}} \int_0^x \left(\int_0^t |g(t,s,y(s))|\,ds\right)^p dt$$

from definition (2.1), for $0 \le x \le b$, we have:

$$\int_0^x \int_0^t \left(\int_0^s |g(s,\tau,y(\tau))|\,d\tau\right)^2 ds\,dt = \int_0^x (x-t) \left(\int_0^t |g(t,s,y(s))|\,ds\right)^p dt$$

$$= {}_0^x I^2 \left(\int_0^t |g(t,s,y(s))|\,ds\right)^p$$

thus by Lemma 2.2 , it follows that $\int_0^x \left(\int_0^t |g(t,s,y(s))|\,ds\right)^p dt$ is Lebesgue integrable for all $x \in (0,b)$ and by Theorem 2.1 we have $|h_3(x)|^p$ is Lebesgue integrable, so $|h(x)|^p$ is Lebesgue integrable, therefore T maps $L^p(0,b)$ into itself. Now, to prove that T is a contraction mapping on $L^p(0,b)$, let $y_1, y_2 \in L^p(0,b)$ then

$$\|Ty_2 - Ty_1\|^p = \Bigg\| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[f(t, y_2(t)) - f(t, y_1(t))\right] dt$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \int_0^t \left[g(t,s,y_2(s)) - g(t,s,y_1(s))\right] ds\,dt \Bigg\|^p$$

$$\le \frac{2^p}{(\Gamma(\alpha))^p} \int_0^b \Bigg[\left(\int_0^x (x-t)^{\alpha-1} |f(t,y_2(t)) - f(t,y_1(t))|\,dt\right)^p$$

$$+ \left(\int_0^x (x-t)^{\alpha-1} \int_0^t |g(t,s,y_2(s)) - g(t,s,y_1(s))|\,ds\,dt\right)^p \Bigg] dx$$

by using condition (ii) we get

$$\|Ty_2 - Ty_1\|^p \le \frac{2^p}{\Gamma(\alpha)^p}\int_0^b \left[\left(\int_0^x (x-t)^{\alpha-1} Z(t)\,|y_2-y_1|\,dt\right)^p + \left(\int_0^x (x-t)^{\alpha-1} P(t)\,|y_2-y_1|\,dt\right)^p\right]dx$$ by

Hölder's inequality

$$\|Ty_2 - Ty_1\|^p \le \frac{2^p}{\Gamma(\alpha)^p}\int_0^b \left[\left(\int_0^x (x-t)^{q(\alpha-1)}\,dt\right)^{\frac{1}{q}} \left(\int_0^x Z^p(t)|y_2-y_1|^p dt\right)^{\frac{1}{p}}\right]^p dx$$

$$+\frac{2^p}{\Gamma(\alpha)^p}\int_0^b \left[\left(\int_0^x (x-t)^{q(\alpha-1)}\,dt\right)^{\frac{1}{q}} \left(\int_0^x P^p(t)\,|y_2-y_1|^p dt\right)^{\frac{1}{p}}\right]^p dx$$

$$\le \frac{(p-1)^{p-1}\,2^p}{(p\alpha-1)^{p-1}\,\Gamma(\alpha)^p}\left[\int_0^b \left(x^{p\alpha-1}\int_0^x Z^p(t)|y_2-y_1|^p dt\right)dx\right.$$

$$\left.+\int_0^b \left(x^{p\alpha-1}\int_0^x P^p(t)|y_2-y_1|^p dt\right)dx\right] \qquad (3.8)$$

to evaluate the integral in the right hand side of inequality (3.8) , let

$$r(x) = \int_0^x Z^p(t)|y_2-y_1|^p dt \quad \Rightarrow \quad r'(x) = Z^p(x)|y_2-y_1|^p$$

$$w(x) = \int_0^x P^p(t)|y_2-y_1|^p dt \quad \Rightarrow w'(x) = P^p(x)|y_2-y_1|^p$$

hence inequality (3.8) becomes

$$\|Ty_2 - Ty_1\|^p \le \frac{(p-1)^{p-1}\,2^p}{(p\alpha-1)^{p-1}\,\Gamma(\alpha)^p}\left[\int_0^b x^{p\alpha-1} r(x)\,dx + \int_0^b x^{p\alpha-1} w(x)\,dx\right]$$

$$\le \left(\frac{p-1}{p\alpha-1}\right)^{p-1}\frac{2^p}{p\alpha\,\Gamma(\alpha)^p}\left[\left(b^{p\alpha}\int_0^b Z^p(x)|y_2-y_1|^p dx - \int_0^b x^{p\alpha}Z^p(x)|y_2-y_1|^p\,dx\right)\right.$$

$$\left.+\left(b^{p\alpha}\int_0^b P^p(x)|y_2-y_1|^p dx - \int_0^b x^{p\alpha}P^p(x)|y_2-y_1|^p\,dx\right)\right]. \qquad (3.9)$$

Since $\int_0^b x^{p\alpha}Z^p(x)|y_2-y_1|^p\,dx \ge 0$ and $\int_0^b x^{p\alpha}P^p(x)|y_2-y_1|^p\,dx \ge 0$, hence inequality (3.9) becomes

$$\|Ty_2-Ty_1\|^p \le \left(\frac{p-1}{p\alpha-1}\right)^{p-1} \frac{2^p}{\left(p\alpha\,\Gamma(\alpha)\right)^p}\left[\left(b^{p\alpha}\int_0^b z^p(t)\left|y_2-y_1\right|^p dt\right)+\left(b^{p\alpha}\int_0^b P^p(t)\left|y_2-y_1\right|^p dt\right)\right]$$

since $z(x)$ and $P(x)$ are non-negative continuous and bounded functions , therefore

$z(x)\le k_1$, $P(x)\le k_2$ for all $x\in[0,b]$, so

$$\|Ty_2-Ty_1\|^p \le \left(\frac{p-1}{p\alpha-1}\right)^{p-1} \frac{2^p b^{p\alpha}\left(k^p_1+k_2^p\right)}{\left(p\alpha\,\Gamma(\alpha)\right)^p}\left(\int_0^b \left|y_2-y_1\right|^p dt\right)$$

$$\|Ty_2-Ty_1\|^p \le \left(\frac{p-1}{p\alpha-1}\right)^{p-1} \frac{2^p b^{p\alpha}(K_1+K_2)^p}{\left(p\alpha\,\Gamma(\alpha)\right)^p}\left(\int_0^b \left|y_2-y_1\right|^p dt\right)$$

$$\|Ty_2-Ty_1\| \le \frac{2b^{\alpha}(K_1+K_2)}{\Gamma(\alpha)}\sqrt[p]{\frac{(p-1)^{p-1}}{p\alpha\,(p\alpha-1)^{p-1}}}\;\|y_2-y_1\|.$$

From (3.4), T is a contraction mapping on $L^p(0,b)$. Thus T has a fixed point say $y(x)\in L^p(0,b)$ that is $Ty(x)=y(x)$.

Next, we study the asymptotic behavior of solutions for the fractional integro-differential equation (1.1) satisfying (1.2).

Theorem 3.2. Let the functions $f(x,y)$ and $g(x,t,y(t))$ of the fractional integro-differential equation (1.1), satisfy the following conditions:

$$\left|f(x,y(x))\right| \le \left|\phi(x)\right|\left|y(x)\right| \tag{3.10}$$

$$\left|\int_0^x g(x,t,y(t))dt\right| \le \left|\beta(x)\right|\left|y(x)\right| \tag{3.11}$$

where $\phi(x)$ and $\beta(x)$ are continuous functions for all $x\in(0,b)$ as $b\to\infty$, such that

$$\lim_{x\to\infty}\int_0^x (x-t)^{\alpha-1}\left(\left|\phi(x)\right|+\left|\beta(x)\right|\right)dt \le M \quad ,M>0 \tag{3.12}$$

then $|y(x)|$, where $y(x)$ is a solution of fractional integro-differential equation (1.1) satisfying (1.2),

is asymptotic to $K\sum_{i=0}^{n-1}\frac{|l_{i+1}|x^i}{i!}$ as x tends to infinity.

Proof. Any solution of the fractional differential equation (1.1) satisfying (1.2) is defined by

$$y(x)=\sum_{i=0}^{n-1}\frac{l_{i+1}x^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t,y(t))dt+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\int_0^t g(t,s,y(s))\,ds\,dt$$

$$|y(x)|\le\sum_{i=0}^{n-1}\frac{|l_{i+1}|x^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}|f(t,y(t))|dt+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\left|\int_0^t g(t,s,y(s))\,ds\right|dt$$

by using conditions (3.10) and (3.11), we get:

$$|y(x)|\le H(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}|\phi(t)||y(t)|dt+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}|\beta(t)||y(t)|dt \qquad (3.13)$$

where $H(x)=\sum_{i=0}^{n-1}\frac{|l_{i+1}|x^i}{i!}$ is a non-decreasing function. From inequality (3.13) and Gronwall's Lamma

and by using condition (3.12), we obtain:

$$|y(x)|\le H(x)\exp\left(\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\left[\,|\phi(t)|+|\beta(t)|\,\right]dt\right)$$

$$|y(x)|\le H(x)\exp\; M_1$$

where $M_1=\frac{M}{\Gamma(\alpha)}$, M_1 is a positive constant, thus $|y(x)|\le K\;H(x)$ for all $x\ge 0$

, $K=\exp(M_1)$, we have

$$|y(x)|\le K\sum_{i=0}^{n-1}\frac{|l_{i+1}|x^i}{i!}.$$

So $|y(x)|$ has the given asymptotic property, hence the proof is complete.

The following example is an application of the theorem (3.2).

Example. Consider the following fractional integro-differential equation

$$y^{(3.25)}(x)=\frac{e^{-2x}\,y(x)}{1+\cosh(x)}+\int_0^x(e^{-t}-te^{-t})|y(t)|dt \qquad ,\ 3<\alpha=3.25<4, \qquad (3.14)$$

with initial conditions:

$$y(0)=1,\ y'(0)=2,\ y''(0)=3 \text{ and } y'''(0)=4. \tag{3.15}$$

Here we have

$$f(x,y(x))=\frac{e^{-2x}\,y(x)}{1+\cosh(x)} \text{ and } g(x,t,y(t))=\ (e^{-t}-te^{-t})\left|y(t)\right|$$

where $f(x,y(x))$ and $g(x,t,y(t))$ satisfy the conditions (3.10) and (3.11) as follows

$$\left|\frac{e^{-2x}\,y(x)}{1+\cosh(x)}\right| \le \frac{e^{-x}}{1+\cosh(x)}\ \ y(x)$$

$$\left|\int_0^x (e^{-t}-te^{-t})\left|y(t)\right|dt\right| \le x\,e^{-x}\,y(x)\,, \quad \text{(using the integration by parts).}$$

Moreover

$$\int_x^\infty (t-x)^{2.25}\left(\frac{e^{-t}}{1+\cosh(t)}+te^{-t}\right)dt \le \frac{1}{2}\int_x^\infty t^{2.25}e^{-t}dt+\int_x^\infty t^{3.25}e^{-t}dt \le \frac{1}{2}\int_0^\infty t^{2.25}e^{-t}dt+\int_0^\infty t^{3.25}e^{-t}dt\,,$$

by the definition of Gamma function we get

$$\int_x^\infty (t-x)^{2.25}\left(\frac{e^{-t}}{1+\cosh(t)}+te^{-t}\right)dt \le \frac{1}{2}\Gamma(3.25)+\Gamma(4.25)=9.55968\,,$$

therefore by Theorem 3.2 the solution of the fractional integro-differentia equation (3.14) satisfying (3.15) is asymptotic to $H(x)e^{4.4229}$, where $H(x)=1+2x+\frac{3x^2}{2}+\frac{4x^3}{6}$.

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